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EXACT LYAPUNOV EXPONENT FOR INFINITE PRODUCTS OF RANDOM MATRICES

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Abstract

In this work, we give a rigorous explicit formula for the Lyapunov exponent for some binary infinite products of random 2×2 real matrices. All these products are constructed using only two types of matrices, A and B , which are chosen according to a stochastic process. The matrix A is singular, namely its determinant is zero. This formula is derived by using a particular decomposition for the matrix B , which allows us to write the Lyapunov exponent as a sum of convergent series. Finally, we show with an example that the Lyapunov exponent is a discontinuous function of the given parameter.

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1) Introduction

The product of random matrices appears in the study of disordered systems [1] as well as in the context of dynamical systems [2]. The Lyapunov exponents are one of the tools to study these products. They are related to physical quantities in disordered systems [3]. For example, in the tight binding model or Anderson model [3], the localisation length of the wave function is proportional to the inverse of the Lyapunov exponent. In dynamical systems theory, products of random matrices often arise as a non trivial approximation which mimics strongly chaotic behavior in deterministic systems.

In spite of the important and numerous results obtained in the theory of random matrices [4,5,6], there is no general method for calculating the Lyapunov exponents [7,8,9], although for a few examples there is an explicit formula [10,11].

In this paper, we present some examples of products of random matrices, where we were able to determine the Lyapunov exponents as a sum of explicitly convergent series. In all these examples, we deal with infinite binary random products, built with only two types of 2×2 real matrices, A and B which are chosen according to a stochastic Bernoulli or Markovian process and one of the matrices, say A , is singular, i.e. its determinant is zero. Markovian processes are chosen to mimic the correlations existing in dynamical systems exhibiting weak chaos [12]. The key of our study is the use of a particular decomposition when the matrix B is non-singular (but if B is singular, the calculation can be done directly). The Bernoulli case was studied by Pincus [13], who gave asymptotic results for the Lyapunov exponent, but which do not enable an actual computation, see remark 2.3 below (also Derrida and Hilhorst [14] have obtained a similar formula for a particular product). Instead, we give an explicit form of the terms of that series. In some cases, this permits either an explicit summation of the series, or a control of the convergence, therefore leading to approximative results. Furthermore, we could show that the convergence of the series is exponentially fast.

The Markovian products are treated in the same way as the Bernoulli one.

The paper is organized as follows. In section 2, we derive the general formula for the largest Lyapunov exponent γ . Section 3 is devoted to the simple case where B is a singular matrix. In section 4 we perform the decomposition of B which, in turn, leads to the decomposition of B^n and in section 5 we derive the expression of γ . In the last section, we analyze the continuity of γ as function of parameters.

2) Lyapunov exponent, general formula

We consider A and B , 2×2 real matrices, A and B , where A is a singular matrix. Now let \mathcal{P}_N :

$$\mathcal{P}_N = X_N X_{N-1} \cdots X_2 X_1$$

be a binary product of N matrices, where the matrix X_i is either A or B , the choice being made by a stochastic process. In the Bernoulli case, we have $X_i = A$ with probability p ($0 < p < 1$) and $X_i = B$ with probability $q = 1 - p$, ($i \geq 1$). In the Markovian case, the transition probabilities are given by

$$\begin{cases} Pr(X_{n+1} = A/X_n = B) = p_1 \\ Pr(X_{n+1} = B/X_n = A) = p_2 \\ Pr(X_{n+1} = B/X_n = B) = 1 - p_1 = q_1 \\ Pr(X_{n+1} = A/X_n = A) = 1 - p_2 = q_2 \end{cases} \quad (2.1)$$

and

$$\begin{cases} Pr(X_1 = A) = \frac{p_1}{p_1 + p_2} = p_0 \\ Pr(X_1 = B) = \frac{p_2}{p_1 + p_2} = q_0 \end{cases} \quad (2.2)$$

with $0 < p_1 < 1$ and $0 < p_2 < 1$.

By definition, the Lyapunov exponent γ is

$$\gamma = \lim_{N \rightarrow \infty} \frac{1}{N} \log \| \mathcal{P}_N \|. \quad (2.3)$$

γ is independent of the choice of the norm $\| \cdot \|$.

Since the matrix A is singular, it can be written by a change of basis in one of the following form :

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \quad (2.4)$$

or,

$$A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \quad (2.5)$$

If A is of the type (2.5), the Lyapunov exponent is $\gamma = -\infty$. This is straightforward to show, since $A^2 = 0$.

Therefore we suppose, without loss of generality, that A has the form (2.4), i.e. $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$.

If we write B^n in the form $B^n = \begin{pmatrix} b_{11}(n) & b_{12}(n) \\ b_{21}(n) & b_{22}(n) \end{pmatrix}$, then the result obtained by Pincus [13], in the Bernoulli case, follows :

$$\gamma = p \log |\lambda| + \sum_{n=1}^{\infty} p^2 (1-p)^n \log |b_{11}(n)|, \quad (2.6)$$

Where $p^2(1-p)^n$ is the probability to obtain the subproduct AB^nA . Even if (2.6) was proved only in the Bernoulli case [13], the same argument extends to the Markovian case, leading to the following proposition.

Proposition 2.1: Let $\{ \mathcal{P}_N \}$ be an infinite product of random matrices satisfying the Markovian distribution law (2.1) and (2.2), where A is a singular matrix given by (2.4) and B is general.

Then

$$\gamma = \frac{p_1}{p_1 + p_2} \log |\lambda| + \sum_{n=1}^{\infty} p_0 p_1 p_2 q_1^{n-1} \log |b_{11}(n)| \quad (2.7)$$

where $b_{11}(n)$ will be explicitly computed in section 4.

Remark 2.2: The proof of this proposition is analogous to that given in [13] once we notice that the probability to find the product AB^nA is $p_0p_1q_1^{n-1}$ in the Markovian case whereas it is $p^2(1-p)$ in the Bernoulli case.

Remark 2.3: Notice that, in order to give an explicit value for γ , from (2.6) for the Bernoulli case or (2.7) for the markovian case, the key point is the computation of $b_{11}(n)$, a problem which was not addressed in [13]. This is the object of the next sections

According to (2.7), it is possible to study the Bernoulli and the Markovian cases in the same manner, by writing

$$\gamma = Pr(A) \log |\lambda| + L \sum_{n=1}^{\infty} x^{n-1} \log |b_{11}(n)| \quad (2.8)$$

where

$$\begin{cases} Pr(A) = p \\ L = p^2(1-p) \\ x = 1-p \end{cases} \quad (2.9)$$

in the Bernoulli case, and

$$\begin{cases} Pr(A) = \frac{p_1}{p_1 + p_2} \\ L = \frac{p_1^2 p_2}{p_1 + p_2} \\ x = q_1 \end{cases} \quad (2.10)$$

in the Markovian case.

3) The case of two singular matrices

In the case where B is singular, it is easy to calculate $b_{11}(n)$. Indeed, as explained, above we have two different cases,

either

$$B = Q^{-1} \begin{pmatrix} \lambda_b & 0 \\ 0 & 0 \end{pmatrix} Q$$

or,

$$B = Q^{-1} \begin{pmatrix} 0 & \lambda_b \\ 0 & 0 \end{pmatrix} Q$$

where Q is an invertible matrix and Q^{-1} is its inverse.

In the former case $\gamma = -\infty$, by the same argument as when A is given by (2.5).

In the latter case, if Q is written as : $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, then we have, for $n \geq 1$, $b_{11}(n) = \lambda_b^n q_{11} q_{22}$. And it is then easy to obtain γ ; for a Bernoulli product, we have

$$\gamma = \gamma_B = p \log |\lambda| + (1-p) \log |\lambda_b| + p(1-p) \log \left| \frac{b_{11}}{Tr(B)} \right| \quad (3.1)$$

and for a Markovian product, we have

$$\gamma = \gamma_M = \frac{p_1}{p_1 + p_2} \log |\lambda| + \frac{p_2}{p_1 + p_2} \log |\lambda_b| + \frac{p_1 p_2}{p_1 + p_2} \log \left| \frac{b_{11}}{Tr(B)} \right| \quad (3.2)$$

where b_{11} denotes the first entry of the matrix B , and $Tr(B)$ is the trace of B .

In the formulae (2.11) and (2.12), we notice a non-linear term, originating in the non-commutativity of the matrices A and B .

4) Normal form and computation of B^n

When B is non-singular, we introduce a decomposition, which enables us to determine $b_{11}(n)$ for all positive integers n .

Let B be a real, non-singular 2×2 matrix, which we write as

$$B = |\det B| \mathcal{R}(-\varphi) \mathcal{B} \mathcal{R}(\varphi) \quad (4.1)$$

where $\det B$ is the determinant of B , $\mathcal{R}(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$ is a matrix rotation in the plane with an angle φ and $\mathcal{R}(-\varphi)$ is its inverse. The angle φ is determined by

$$\tan(2\varphi) = \frac{b_{22} - b_{11}}{b_{12} + b_{21}} \quad (4.2)$$

where b_{ij} are the entries of B .

We call the matrix \mathcal{B} , the normal form of B . We have four different types of normal forms \mathcal{B} , depending on the sign of the determinant and on the eigenvalues of B , which can be real, non-degenerated eigenvalues, real degenerated eigenvalues and conjugate complex eigenvalues.

We now define the quantity :

$$\rho = \sqrt{\left| \frac{l_2}{l_1} \right|} \quad (4.3)$$

where

$$l_1 = b_{12} \sin^2(\varphi) - b_{21} \cos^2(\varphi) + \frac{1}{2}(b_{11} - b_{22}) \sin(2\varphi)$$

and

$$l_2 = b_{12} \cos^2(\varphi) - b_{21} \sin^2(\varphi) - \frac{1}{2}(b_{11} - b_{22}) \sin(2\varphi)$$

ρ will be used in the expression of \mathcal{B} .

Remark 4.1: Replacing B by $C = \frac{1}{\det B} B$, accroding to (2.3), γ is shifted by an additive constant, $\log |\det B|$. Therefore, we can assume without loss of generality, that $|\det B| = 1$.

We now consider four different cases.

Case I : B is hyperbolic symplectic

The eigenvalues of B are real non-degenerated and have the same sign, $\lambda_{1,2} = \epsilon \exp \pm \sigma$ with σ non zero and $\epsilon = 1$ or -1 ; then, we get

$$\mathcal{B} = \epsilon \begin{pmatrix} \cosh \sigma & \rho \sinh \sigma \\ \frac{1}{\rho} \sinh \sigma & \cosh \sigma \end{pmatrix} \quad (4.4)$$

Case II : B is hyperbolic, non-symplectic

In this case the eigenvalues of B are also real, non-degenerated but they have opposite signs, namely $\lambda_1 = \exp \sigma$ and $\lambda_2 = -\exp -\sigma$ or $-\lambda_1$ and $-\lambda_2$.

Although we have two possible forms for \mathcal{B} , we can deduce one from the other by changing the sign of φ . We therefore retain only one form for \mathcal{B} :

$$\mathcal{B} = \begin{pmatrix} \sinh \sigma & \rho \cosh \sigma \\ \frac{1}{\rho} \cosh \sigma & \sinh \sigma \end{pmatrix} \quad (4.5)$$

Case III : B is parabolic

The eigenvalues of B are given by $\lambda_{1,2} = \epsilon$, ($\epsilon = 1$ or -1) and we have either

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & \epsilon l_2 \\ 0 & 1 \end{pmatrix}, \quad (4.6)$$

or

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & 0 \\ -\epsilon l_1 & 1 \end{pmatrix} \quad (4.7)$$

Notice that we can pass from (4.7) to (4.6) by a rotation in the plane with an angle $\frac{\pi}{2}$, which is equivalent to a change of the sign of φ . In the following, we define the normal form of a parabolic matrix as given by (4.6).

Case IV : B is elliptic

The eigenvalues of B are complex conjugate, $\lambda_{1,2} = \exp \pm i\sigma$, in which case

$$\mathcal{B} = \begin{pmatrix} \cos \sigma & -\rho \sin \sigma \\ \frac{1}{\rho} \sin \sigma & \cos \sigma \end{pmatrix}. \quad (4.8)$$

Therefore, if B is a non-singular matrix, we can write it in the form given by (4.1) where \mathcal{B} is given by one of the expressions (4.4), (4.5), (4.6) or (4.8).

We can now easily compute B^n in each of the previous cases. We summarize the result in the following proposition.

Proposition 4.2: Let B be a non-singular matrix which a normal form \mathcal{B} defined by (4.1)

Then

$$B^n = \mathcal{R}(-\varphi)\mathcal{B}^n\mathcal{R}(\varphi). \quad (4.9)$$

If B is symplectic hyperbolic, given by (4.4), then

$$\mathcal{B}^n = \epsilon^n \begin{pmatrix} \cosh n\sigma & \rho \sinh n\sigma \\ \frac{1}{\rho} \sinh n\sigma & \cosh n\sigma \end{pmatrix}. \quad (4.10)$$

If B is non-symplectic hyperbolic, given by (4.5), then either

$$\mathcal{B}^n = \begin{pmatrix} \cosh n\sigma & \rho \sinh n\sigma \\ \frac{1}{\rho} \sinh n\sigma & \cosh n\sigma \end{pmatrix} \quad (4.11)$$

if n is odd; or

$$\mathcal{B}^n = \begin{pmatrix} \sinh n\sigma & \rho \cosh n\sigma \\ \frac{1}{\rho} \cosh n\sigma & \sinh n\sigma \end{pmatrix} \quad (4.12)$$

if n is even.

If B is parabolic, given by (4.6), then

$$\mathcal{B}^n = \epsilon^n \begin{pmatrix} 1 & n\epsilon^n l_2 \\ 0 & 1 \end{pmatrix}. \quad (4.13)$$

If B is elliptic, given by (4.8), then

$$\mathcal{B}^n = \begin{pmatrix} \cos n\sigma & -\rho \sin n\sigma \\ \frac{1}{\rho} \sin n\sigma & \cos n\sigma \end{pmatrix} \quad (4.14)$$

5) Lyapunov Exponent : explicit formulae

We are now ready to perform the analysis of the formula (2.8) in the case where the matrix B is non-singular.

As previously, without loss of generality, we suppose that $|\det B| = 1$.

We now give the value of the largest Liapunov exponent in each of the four cases treated in the previous section.

Case I : B is an hyperbolic symplectic matrix

For $n \geq 1$, we have

$$b_{11}(n) = \epsilon^n [\cosh(n\sigma) + \cosh(\alpha) \sinh(n\sigma) \sin(2\varphi)] \quad (5.1)$$

where

$$\cosh \alpha = \frac{1}{2} \left(\rho + \frac{1}{\rho} \right) \quad (5.2)$$

In (5.1), we can take $\sigma > 0$. Indeed, if σ is negative, we change the sign of φ and use $-\sigma$ instead of σ .

Notice first that if it exists an integer $n_0 \geq 1$ such that $b_{11}(n_0) = 0$ then $\gamma = -\infty$.

The condition $b_{11}(n_0) = 0$ is equivalent to

$$\cosh \alpha = -\frac{1}{\sin(2\varphi) \tanh(n_0\sigma)}$$

and therefore we can construct some products for wich $\gamma = -\infty$.

On the contrary, if we suppose that for all integers $n \geq 1$, $b_{11}(n) \neq 0$, and we define the quantities

$$\delta = 1 + \sin(2\varphi) \cosh(\alpha) \quad (5.3)$$

and

$$\tau = \frac{1 - \sin(2\varphi) \cosh(\alpha)}{1 + \sin(2\varphi) \cosh(\alpha)} \quad (5.4)$$

We will use τ and δ in the expression of γ .

We now distinguish two cases.

a) $\delta = 0$:

A straithforward calculation gives

$$\gamma = \gamma_B = p \log |\lambda| - (1-p)\sigma \quad (5.5)$$

for the Lyapunov exponent in the Bernoulli case, and

$$\gamma = \gamma_M = \frac{p_1}{p_1 + p_2} \log |\lambda| - \frac{p_2}{p_1 + p_2} \sigma \quad (5.6)$$

for the Markovian case.

b) $\delta \neq 0$:

Depending on whether τ is zero or not, we have different expressions for γ .

i) $\tau = 0$: we easily obtain the largest Lyapunov exponent for a Bernoulli product:

$$\gamma = \gamma_B = p \log |\lambda| + (1 - p) \sigma, \quad (5.7)$$

and for a Markovian product:

$$\gamma = \gamma_M = \frac{p_1}{p_1 + p_2} \log |\lambda| + \frac{p_2}{p_1 + p_2} \sigma \quad (5.8)$$

ii) $\tau \neq 0$: Here we obtain γ as a convergent series given by

$$\begin{aligned} \gamma &= Pr(A) \log |\lambda| + \frac{L}{(1-x)^2} \sigma \\ &+ \frac{L}{1-x} \log \frac{|\delta|}{2} + L \sum_{n=1}^{\infty} x^{n-1} \log |1 + \tau e^{-2n\sigma}|. \end{aligned} \quad (5.9)$$

where L is defined by (2.9) and (2.10).

Notice that if γ_N is the sum of the N -th first terms in (5.9) of γ , the error is

$$|\gamma - \gamma_N| \leq |\tau| \exp(-2(N+1)\sigma) L x^{N+1} \quad (5.10)$$

and therefore it is exponentially small since $0 < x < 1$.

Case II : B is an hyperbolic no symplectic matrix

This case is similar to the previous one with only a slight difference concerning (5.9). By applying the decomposition given by (4.1) we obtain

$$b_{11}(n) = \frac{1 + \sin(2\varphi) \cosh(\alpha)}{2} e^{n\sigma} + (-1)^n \frac{1 - \sin(2\varphi) \cosh(\alpha)}{2} e^{-n\sigma}. \quad (5.11)$$

Again, we can suppose that $\sigma > 0$ without loss of generality. And, as above, if there exists an $n_0 \geq 1$ such that $b_{11}(2n_0) = 0$ or $b_{11}(2n_0 - 1) = 0$, then $\gamma = -\infty$.

Recall that δ and τ are defined by (5.3) and (5.4).

If $\delta = 0$ or if $\delta \neq 0$ but $\tau = 0$, the largest Lyapunov exponent is given by the same expressions as in the previous case: (5.5) and (5.6), or (5.7) and (5.8).

Instead, if $\delta \neq 0$ and $\tau \neq 0$, we obtain

$$\begin{aligned} \gamma &= Pr(A) \log |\lambda| + \frac{L}{(1-x)^2} \sigma + \frac{L}{1-x} \log \left| \frac{\delta}{2} \right| \\ &+ L \sum_{n=1}^{\infty} x^{n-1} \log |1 + (-1)^n \tau e^{-2n\sigma}|. \end{aligned} \quad (5.12)$$

This serie is convergent, indeed we have

$$|\gamma - \gamma_N| \leq Lx^{N+1} \Delta \quad (5.13)$$

where

$$\Delta = \max(|\log |1 + (-1)^{N+1} e^{-2(N+1)\sigma}||, |\log |1 + (-1)^{N+2} e^{-2(N+2)\sigma}||). \quad (5.14)$$

Case III : B is a parabolic matrix

The normal form of B is

$$\mathcal{B} = \epsilon \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and thus, for $n \geq 1$,

$$b_{11}(n) = \epsilon^n [1 + \frac{1}{2} n b \sin(2\varphi)]. \quad (5.15)$$

We suppose that $b_{11} \neq 0$ for all integers $n \geq 1$. The trivial case where $\sin(2\varphi) = 0$ corresponds to an infinite product of diagonal and triangular matrices. In this case,

$$\gamma = Pr(A) \log |\lambda|.$$

When, instead, $\sin(2\varphi) \neq 0$, we have, for a Bernoulli product:

$$\gamma = p \log |\lambda| + p(1-p) \log \left| \frac{b \sin(2\varphi)}{2} \right| + p^2(1-p) \sum_{n=1}^{\infty} (1-p)^{n-1} \log \left| n + \frac{1}{nb \sin(2\varphi)} \right| \quad (5.16)$$

and, for a Markovian product,

$$\gamma = \frac{p_1}{p_1+p_2} \log |\lambda| + \frac{p_1 p_2}{p_1+p_2} \log \left| \frac{b \sin(2\varphi)}{2} \right| + \frac{p_1 p_2}{p_1+p_2} \sum_{n=1}^{\infty} (1-p_1)^{n-1} \log \left| n + \frac{1}{b \sin(2\varphi)} \right|. \quad (5.17)$$

These series are convergent since we have

$$|\gamma - \gamma_N| \leq \left| \log \left| 1 + \frac{2}{nb \sin(\varphi)} \right| \right| \leq \left| \frac{1}{1-x} + \frac{N+1}{(1-x)^2} \right| x^N \quad (5.18)$$

and $0 < x < 1$.

Case IV : B is an elliptic matrix

In this case we also obtain series for γ , but it is very difficult, in general, to evaluate the rest of the corresponding partial sums. Indeed, in this situation, as expected, the summability of the series in the expression of γ is related to the arithmetic properties of σ , when the latter is irrational (*mod* 2π). On the another hand, if σ is rational (*mod* 2π), γ may be given as a sum of a finite number of terms.

As above, by using the decomposition (4.1), we obtain

$$b_{11}(n) = \cos(n\sigma) + \sinh(\alpha) \sin(n\sigma) \sin(2\varphi) \quad (5.19)$$

where

$$\sinh(\alpha) = \frac{1}{2} \left(\rho - \frac{1}{\rho} \right) \quad (5.20)$$

with $\alpha \in \mathbb{R}$.

If $b_{11}(n_0) = 0$ for some positive integer n_0 , then $\gamma = -\infty$.

Suppose now that $b_{11}(n) \neq 0$, for all positif integers n , and σ is rational (*mod* 2π), i.e. $\sigma = \frac{r}{s} 2\pi$ where $r, s \in \mathbb{N}^*$ are irreducible. Thus

$$\gamma = p \log |\lambda| + \frac{p^2}{1 - (1-p)^s} \sum_{j=1}^{s-1} (1-p)^j \log |b_{11}(j)| \quad (5.21)$$

for a Bernoulli product,
and

$$\gamma = \frac{p_1}{p_1 + p_2} \log |\lambda| + \frac{p_1^2 p_2}{(p_1 + p_2)(1 - (1 - p_1)^s)} \sum_{j=1}^{s-1} (1 - p_1)^{j-1} \log |b_{11}(j)| \quad (5.22)$$

where

$$b_{11}(j) = \cos\left(\frac{2jr}{s}\pi\right) + \sinh \alpha \sin(2\varphi) \sin\left(\frac{2jr}{s}\pi\right) \quad (5.23)$$

for a Markovian product.

6) One parameter family of products: an example.

One may wonder if a limit procedure may permit the computation of γ in the elliptic case for irrational σ from the corresponding formulae (5.21) or (5.23), when σ is rational.

More generally, the continuity of the largest Lyapunov exponent as a function of the amplitude of the disorder in disordered systems is known as being an important issue. In this section, we illustrate the results of the previous sections by giving an example where, indeed, γ is a discontinuous function of a parameter α .

We consider the family $\{P_\infty(\alpha) = (A, B(\alpha), p_1, p_2), \alpha \in \mathbb{R}^{+*}, p_1 > 0, p_2 > 0\}$ of infinite Markovian products of the singular matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and a family $B(\alpha)$ of matrices, depending on a parameter α . p_1 and p_2 are the transition probabilities defined by (2.1).

If $p_1 + p_2 = 1$, then we recover the case of a Bernoulli product.
 $B(\alpha)$ is defined by

$$B(\alpha) = \mathcal{R}(-\varphi) \begin{pmatrix} \cosh \sigma_0 & \rho \sinh \sigma_0 \\ \frac{1}{\rho} \sinh \sigma_0 & \cosh \sigma_0 \end{pmatrix} \mathcal{R}(\varphi) \quad (6.1)$$

where $\varphi = -\frac{\pi}{4}$, $\sigma > 0$ is a fixed parameter; and $\rho = e^\alpha$ with $\alpha \in \mathbb{R}^{+*}$.

The condition $b_{11}(\alpha) = 0$ gives us

$$\alpha = \alpha_n = -\text{Argch}(\coth(n\sigma)) \quad (6.2)$$

and $\gamma(\alpha_n) = -\infty$.

Now for all $\alpha \in \mathbb{R}^{+*}$, we define

$$\begin{aligned} f(\alpha) = \gamma(\alpha) &= \frac{p_2}{p_1+p_2} \sigma_0 + \frac{p_1 p_2}{p_1+p_2} \log\left(\frac{\cosh \alpha - 1}{2}\right) \\ &+ \frac{p_1^2 p_2}{p_1+p_2} \sum_{n=1}^{\infty} (1-p_1)^{n-1} \log\left(1 + \frac{\cosh \alpha + 1}{\cosh \alpha - 1} e^{-2n\sigma_0}\right) \end{aligned} \quad (6.3)$$

Since the series in (5.3) is sommable, then $f(\alpha)$ is defined and continuous.

But since, for $\alpha \neq \alpha_n$, we have

$$\gamma(\alpha) = f(\alpha), \quad (6.4)$$

We obtain

$$\lim_{\alpha \rightarrow \alpha_n} \gamma(\alpha) = f(\alpha_n) \neq \gamma(\alpha_n) \quad (6.5)$$

for all $n \in \mathbb{N}^*$. Thus $\gamma(\alpha)$ is a discontinuous function of α , for each $\alpha = \alpha_n$, $n \in \mathbb{N}$.

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